Anisotropic spectra of acoustic turbulence

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We found universal anizopropic spectra of acoustic turbulence with the linear dispersion law $\omega(k)=ck$ within the framework of generalized kinetic equation which takes into account the finite time of three-wave interactions. This anisotropic spectra can assume both scale-invariant and non-scale-invariant form. The implications for the evolution of the acoustic turbulence with nonisotropic pumping are discussed. The main result of the article is that the spectra of acoustic turbulence tend to become more isotropic.

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I. INTRODUCTION AND GENERAL DISCUSSION

Wave turbulence, which describes the behavior of a spatially homogeneous field of random dispersive waves, has led to spectacular success in our understanding of spectral energy transfer processes in plasmas, oceans, and planetary atmospheres [1]. In the case of small level of nonlinearity ϵ (for example, for the surface waves this is the ratio of the wave amplitude h to the wavelength $\lambda, \epsilon=h/\lambda$), there is a consistent description of the weak wave turbulence in terms of so called *kinetic equation* (KE) which describes the energy transfer due to interactions of three (in some cases four) waves with the conservation of energy and momenta

$$\omega(\boldsymbol{k})\pm\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)=0,\quad \boldsymbol{k}\pm\boldsymbol{k}_1-\boldsymbol{k}_2=\boldsymbol{0}. \tag{1}$$

These equations on a classical language are a condition of time-space resonance. Equations (1) show that acoustic waves are special: interacting waves have to have parallel wave vectors

$$\boldsymbol{k}\|\boldsymbol{k}_1\|\boldsymbol{k}_2, \tag{2}$$

and therefore the interacting acoustic waves are foliated into noninteracting rays of waves propagating in different directions. Immediately a few important questions arise.

(1) What are the mechanisms responsible for an energy redistribution between different rays?

(2) How does the energy become shared between neighboring rays?

(3) Does the energy tend to diffuse away from the ray with maximum energy or can it focus onto that ray? In the latter case, one might argue that shock formation may again become the relevant process especially if the energy should condense on rays with very different directions.

(4) Is the approximation of the KE adequate for a description of acoustic turbulence even in the case of small nonlinearity?

The positive answer on the last question is problematic [1]. Approximation of KE is based on the randomization of phases of noninteracting waves leading to the Gaussian statistics and requires weakness of the interaction $\epsilon^2\ll 1$ to ensure the closeness the wave statistics to Gaussianity. This happens in some physical situations, but not for acoustic waves. The physical reason is the above described foliation: for each particular (noninteracting) ray of waves one may pass into comoving (with the sound velocity c) reference system in which all the waves are in the rest and therefore their interaction time goes to infinity. Therefore for any small level of nonlinearity the waves have enough time for the finally large deviation of phases from the random Gaussian distribution. This is exactly the reason why in one dimensional case one has a creation of shock waves (which may be described by the Burgers equation) for any (small) level of nonlinearity. If really acoustic waves tend to focus, the approximation of KE is problematic even qualitatively. In this paper we will show that this is not the case and therefore the approximation of KE may serve at least as a starting point to describe nondispersive acoustic turbulence.

The answers on the first three questions for weakly dispersive acoustic waves was done a long ago by one of us (V.L.) and G. Falkovich [2]. We showed that weakly dispersive waves, say with the dispersion law

$$\omega(k)\propto k^{1+\delta},\quad \delta\ll 1, \tag{3}$$

really tends to focus. Namely, the isotropic solution of the KE

$$n_0(k)=ak^{-9/2}\quad (a \text{ is a dimensional constant}), \tag{4}$$

found by Zakharov and Sagdeev in 1970 [3] is unstable in the sense that the anisotropic solutions of the KE found in Ref. [2]:

$$n_{lp}(k)\propto\frac{(kL)^{\delta(l+p)/2}}{k^{9/2}}\left[1-\frac{\boldsymbol{k}\cdot\boldsymbol{n}}{k}\right]^l\left[l+\frac{\boldsymbol{k}\cdot\boldsymbol{n}}{k}\right]^p \tag{5}$$

increases with k from the pumping scale $1/L$ toward the depth of the inertial interval $kL\gg 1$ faster than the isotropic one (4). In Eq. (5) $\boldsymbol{n}$ is the unit vector which is determined by anisotropy of the pumping. Observe that the ratio

$$\frac{n_{lp}(\boldsymbol{k})}{n_0(k)}\propto(kL)^{\delta(l+p)/2} \tag{6}$$

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increases faster for more anisotropic shapes (larger l and p) and for more dispersive waves with bigger δ. In this sense the nondispersive waves with $\delta=0,\omega(k)\propto k$ are marginal, $n_{lp}(\boldsymbol{k})/n_0(k)\propto k^0=$const and any angle distribution $n(\boldsymbol{k})$ is a solution of the KE at $\delta=0$. Clearly this is a consequence of the foliation described above. However, physical intuition tells us that there should be some mechanism of redistributing energy between neighboring rays even for fully nondispersive waves. Indeed, the fact that all interacting waves have parallel wave vectors follows from the time-space resonance conditions (1). These conditions are valid with some accuracy which is determined by the life time of the waves and therefore neighboring rays can really interact. In our recent paper [4] we generalized the KE for acoustic waves to account the finite width of the resonances. Generalized kinetic equation (GKE) for the "occupation numbers of waves" $n(\boldsymbol{k},t)$ has the form

$$\frac{\partial n(\boldsymbol{k},t)}{\partial t}=\mathrm{St}_k(\{n(\boldsymbol{k}',t)\}), \qquad (7)$$

where the collision term $\mathrm{St}_k(\{n(\boldsymbol{k}',t)\})$ is a functional of the occupation numbers $n(\boldsymbol{k}',t)\}$ with all wave vectors $\boldsymbol{k}'$ but at the same moment of time t [which for the shortness we will skip from the arguments: $n(\boldsymbol{k}',t)\Rightarrow n(\boldsymbol{k}')$]. The collision term for GKE is very similar to that for the KE: it is proportional to the square of the amplitude of three wave interactions $V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)$, bilinear in $n(\boldsymbol{k}')$ and actually contains one three-dimensional integration

$$\mathrm{St}_k(\{n(\boldsymbol{k}')\})=\int\frac{d\boldsymbol{k}_1 d\boldsymbol{k}_2}{(2\pi)^3}\Gamma_{k12}\Bigg\{\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\frac{1}{2}\frac{|V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)|^2[n(\boldsymbol{k}_1)n(\boldsymbol{k}_2)-n(\boldsymbol{k})[n(\boldsymbol{k}_1)+n(\boldsymbol{k}_2)]]}{[\omega(\boldsymbol{k})-\omega(\boldsymbol{k}_1)-\omega(\boldsymbol{k}_2)]^2+\Gamma_{k12}^2}$$
$$+\delta(\boldsymbol{k}+k_1-\boldsymbol{k}_2)\frac{|V(\boldsymbol{k}_2,\boldsymbol{k}_1,\boldsymbol{k})|^2[n(\boldsymbol{k}_2)[n(\boldsymbol{k}_1)+n(\boldsymbol{k})]-n(\boldsymbol{k})n(\boldsymbol{k}_1)]}{[\omega(\boldsymbol{k})+\omega(\boldsymbol{k}_1)-\omega_0(\boldsymbol{k}_2)]^2+\Gamma_{k12}^2}\Bigg\}. \qquad (8)$$

This collision term differs from that of the KE in the finite width $\Gamma_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}$ of the resonances (1):

$$\Gamma_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}=\gamma(\boldsymbol{k})+\gamma(\boldsymbol{k}_1)+\gamma(\boldsymbol{k}_2), \qquad (9)$$

where $\gamma(\boldsymbol{k})$ is the damping of monochromatic wave with given $\boldsymbol{k}$. This allows interaction of waves from different, but neighboring rays. We will see that for small nonlinearity, $\epsilon^2\ll 1$ the characteristic angle of interaction is small: $\Delta\theta_k \ll\pi$. This helps us to significantly simplify the collision integral.

The paper is written as follows. In Sec. II we simplify the collision term of the GKE by using the differential in angle approximation. Observe from Eqs. (22), (29), and (30) below that the collision term St_k in the differential approximation has just one-dimensional k integration along the ray with direction of $\boldsymbol{k}$ and operators of differentiation in two orthogonal directions. Note, that Eq. (30) written for distributions $n(\boldsymbol{k})$ with characteristic angle width much larger than the interaction angle. This equation has an isotropic solution $n_0(k)\propto k^{-9/2}$ which coincides with the solution (4) of the KE.

In Sec. III we linearize differential form of the GKE assuming that the deviation $n_1(\boldsymbol{k})$ of the distribution $n(\boldsymbol{k})$ from isotropic solution $n_0(k)$ is small and expand

$$n_1(\boldsymbol{k})\equiv n(k,\cos\theta), \quad \cos\theta=\boldsymbol{k}\cdot\boldsymbol{n}/k$$

into series of Legendre polynomials $P_\ell(\cos\theta)$,

$$n_1(k,\cos\theta)=\sum_{\ell=1}^{\infty} f_\ell(k)P_\ell(\cos\theta). \qquad (10)$$

After that our problem foliates into set of decoupled equations for the Legendre polynomial with a given order ℓ. In such a way we reduce the dimensionality of the problem to dimension one. Now the unknown function $f_\ell(k)$ depends only on one variable k and the corresponding collision terms involve only one-dimensional k integration. Our observation is that the equations for different P_ℓ involve only one combination of the parameters, namely, $\epsilon^2\ell(\ell+1)$. In Sec. III C we found scale-invariant solutions of these equations;

$$f_\ell(k)\propto\frac{1}{k^{x_\ell}}, \quad x_\ell=6+\frac{\ln[\epsilon^2\ell(\ell+1)B]}{\ln(k_* L)}, \qquad (11)$$

which are valid for the region of parameters where $5<x_\ell<6$. In Eq. (11) B is some number of order 1, L is outer scale of turbulence and k_* is the wave vector for which $\gamma(k_*)$ is about the smallest frequency of waves in the system $\simeq c/L$. Note that the isotropic solution (4) $n_0(k)\propto k^{-9/2}$ and therefore when kL increases the ratio $f_\ell(k)/n_0(k)$ decreases at least as $1/\sqrt{kL}$. It means that in a cascade of energy transfer from anisotropic region of pumping down to the depth of the inertial interval energy tends to diffuse between all the rays and asymptotically, in the limit $k\to\infty$ acoustic turbulence become fully isotropical. This phenomenon of isotropization of non-dispersive acoustic turbulence contrasts with self-focusing of weakly dispersive acoustic turbulence discovered in Ref. [2], and see Eqs. (3), (5), (6). Note also that for nondispersive waves the rate of isotropization, i.e., the rate of decreasing the ratio $f_\ell(k)/n_0(k)$, depends only on the combination $\epsilon^2\ell(\ell+1)$ and increases both with ϵ^2 and ℓ.

In Sec. IV we found nonscale invariant solution of the linearized equations for $f_\ell(k\Lambda)$ which depends on some characteristic length Λ:

$$f_\ell(k\Lambda)\sim\sqrt{k\Lambda}\,\exp\left[-\frac{1}{2}\sqrt{\epsilon^2\ell(\ell+1)C}(k\Lambda)^2\right]. \qquad (12)$$

Here C is some constant of order unity. Solution (12) is valid for $k\Lambda > 1$, the value of Λ has to be found from a matching of this solution with a solution for smaller k. Again, the rate of isotropization depends only on the combination $\epsilon^2\ell(\ell+1)$ and increases both with ϵ^2 and ℓ.

Note that the choice between two found solutions is a delicate issue and depends on the various parameters of the problem: ϵ^2 and $\ell(\ell+1)$ separately, on the value of underground dispersion of the waves, etc. We do not think that it is reasonable to study this question in general, without referring to a specific physical realization of the acoustic turbulence.

The main qualitative message of this paper is that in spite of anisotropic pumping at large scales L acoustic turbulence became to be more and more isotropic with increasing of its wave vector. The rate of isotropization increases with increasing the level of nonlinearity ϵ^2 and depends on the characteristic angle of the distribution $\Delta\theta \simeq \pi/\ell$ as $1/(\Delta\theta)^2$.

II. DIFFERENTIAL APPROXIMATION OF THE GENERALIZED KINETIC EQUATION

Our starting point is the generalized kinetic equation, GKE (7), (8) which describes the interaction of neighboring rays due to the finite width of the three-wave resonance (9), determined by the damping increment of an individual wave with given $k,\lambda(k)$. The value of $\gamma(k)$ was calculated in Ref. [4]:

$$\gamma(\boldsymbol{k}) = \nu k^2, \quad \nu \simeq \frac{A^2 N}{4\pi c}, \tag{13}$$

where ν is the effective viscosity, N is the total number of the waves in the system

$$N = \int_{1/L}^{\infty} n(q) q^2 dq, \tag{14}$$

and $A \simeq \sqrt{c/\rho}$ characterizes the three-wave interaction amplitude

$$V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = A\sqrt{k k_1 k_2}. \tag{15}$$

Here we accounted that the interaction is dominant by the interaction of almost collinear wave vectors and neglected the angular dependence of $V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)$.

Integrating Eq. (8) with the help of the δ functions, one has

$$\mathrm{St}_{\boldsymbol{k}}(\{n(\boldsymbol{k}')\}) = \frac{A^2 k}{(2\pi)^3}\left[\frac{1}{2}\int d\boldsymbol{k}_1 k_1 (k-k_1)\frac{N_-\Gamma_-}{\Omega_-^2+\Gamma_-^2} + \int d\boldsymbol{k}_1 k_1 (k+k_1)\frac{N_+\Gamma_+}{\Omega_+^2+\Gamma_+^2}\right]. \tag{16}$$

Here we have used the short-hand notations

$$\Gamma_\pm = \nu[k^2 + k_1^2 + |\boldsymbol{k}\pm\boldsymbol{k}_1|^2],$$

$$N_\pm = n(\boldsymbol{k}\pm\boldsymbol{k}_1)[\boldsymbol{n}(k_1) \pm n(\boldsymbol{k})] - n(\boldsymbol{k})n(\boldsymbol{k}_1), \tag{17}$$

$$\Omega_\pm = c(k \pm k_1 - |\boldsymbol{k}\pm\boldsymbol{k}_1|).$$

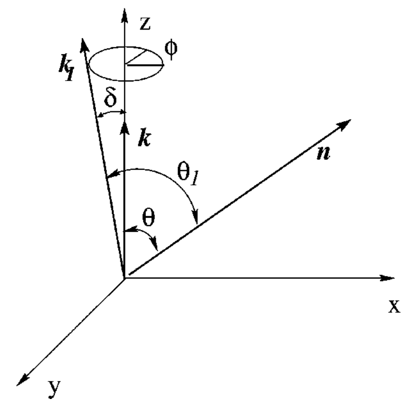

FIG. 1. The coordinate system used in calculation of the integrals in Eq. (7). Vector $\boldsymbol{n}$ denotes the physically unique direction. The axis $\boldsymbol{z}$ is along $\boldsymbol{k}$, the axis $\boldsymbol{x}$ is in the plane $(\boldsymbol{k},\boldsymbol{n})$. The angles between vector $\boldsymbol{n}$ and vectors $\boldsymbol{k}$ and $\boldsymbol{k}_1$ are denoted by θ and θ_1, respectively. δ is the small angle between $\boldsymbol{k}$ and $\boldsymbol{k}_1$. The azimuthal angle of $\boldsymbol{k}_1$ is denoted by ϕ.

Under the above assumption of dominance of the collinear interaction one may take $\mathbf{k}_1 \| \boldsymbol{k}$ in $\Gamma_\pm$ to get

$$\Gamma_\pm = 2\nu(k^2 + k_1^2 \pm k k_1). \tag{18}$$

However, we cannot use the same approximation in the equations for $\Omega_\pm$ and $N_\pm$. To understand this let us have a look at the angular integrals in Eq. (16) at given value of k_1. It is clear that the main contribution to the integrals comes from the region where $|\Omega_\pm|$ varies from 0 to a value of order $\gamma(k)$. Within this region $\Gamma_\pm$ is constant (18) with accuracy of order $\gamma(k)/ck \sim \epsilon^2$. Similar considerations show that we have to account a variation of $N_\pm$ within the region of angular integration in Eq. (16).

To evaluate the integrals in Eq. (16) one has to establish the relations between different wave vectors appearing in the equation. Let $\boldsymbol{n}$ be some physically unique direction and assume for simplicity an axial symmetry around $\boldsymbol{n}$. The vectors $\boldsymbol{k}$ and $\boldsymbol{k}_1$ are almost collinear with a small angle δ between them and arbitrary oriented. Then we introduce a Cartesian coordinate system with the axis $\boldsymbol{z}$ along $\boldsymbol{k}$ (Fig. 1).

Given the geometry, we now derive a differential approximation for the anisotropic GKE. To this end, consider Eq. (17) and expand $\Omega_\pm$ and $N_\pm$ in δ. Clearly,

$$\Omega_\pm \simeq \delta^2 \Omega''_\pm, \quad \Omega''_\pm \equiv \pm\frac{ckk_1}{2(k\pm k_1)}, \tag{19}$$

whereas $N_\pm$ may be written as

$$N_\pm = N_\pm^{(0)} + N_\pm^{(1)} + N_\pm^{(2)}. \tag{20}$$

Here

$$N_\pm^{(0)} = n_{\boldsymbol{k}\pm\boldsymbol{q}}(n_{\boldsymbol{q}} \pm n_{\boldsymbol{k}}) - n_{\boldsymbol{k}} n_{\boldsymbol{q}}, \tag{21}$$

with $\boldsymbol{q} \equiv \boldsymbol{k}k_1/k$. Terms with $N_\pm^{(1)} \propto \delta\cos\theta$ or $\delta\sin\theta$ and disappear after integration over ϕ, while $N_\pm^{(2)} = \delta^2 N''_\pm$. After (free) integration over ϕ, $N''_\pm$ reads

$$N''_{\pm}=\frac{1}{4}[(n_{\boldsymbol q}\pm n_{\boldsymbol k})\nabla_{\perp}^2 n_{\boldsymbol{k}\pm\boldsymbol{k}_1}+(n_{\boldsymbol{k}\pm\boldsymbol{q}}-n_{\boldsymbol k})\nabla_{\perp}^2 n_{\boldsymbol{k}_1} \pm 2(\nabla_{\perp} n_{\boldsymbol{k}_1})\cdot(\nabla_{\perp} n_{\boldsymbol{k}\pm\boldsymbol{k}_1})], \tag{22}$$

where

$$\nabla_{\perp}\equiv\sin\theta\frac{\partial}{\partial\cos\theta_1} \tag{23}$$

and θ_1 is the angle between vectors $\boldsymbol{k}_1$ and $\boldsymbol{n}$; derivatives are taken at $\theta_1=\theta$.

Note, that all the δ dependence of the integrand in Eq. (16) is hidden in $\Omega_{\pm}$ and $N_{\pm}$ and therefore the integrals may be factorized. Using smallness of δ we write explicitly $d\boldsymbol{k}_1=\pi q^2 dq\, d\delta^2$ and consider the integrals over δ^2:

$$I_{\pm}=\pi\int d\delta^2\frac{N_{\pm}\Gamma_{\pm}}{\Omega_{\pm}^2+\Gamma_{\pm}^2}=I_{\pm}^{(0)}+I_{\pm}^{(2)}, \tag{24}$$

where

$$I_{\pm}^{(0)}=\pi N_{\pm}^{(0)}\Gamma_{\pm}\int_0^{\infty}\frac{d\delta^2}{\delta^4(\Omega''_{\pm})^2+\Gamma_{\pm}^2}=\frac{\pi^2}{2}\frac{N_{\pm}^{(0)}}{\Omega''_{\pm}}, \tag{25}$$

$$I_{\pm}^{(2)}=\pi N''_{\pm}\Gamma_{\pm}\int_0^{b}\frac{\delta^2 d\delta^2}{\delta^4(\Omega''_{\pm})^2+\Gamma_{\pm}^2}=\frac{\pi}{2}\frac{N''_{\pm}\Gamma_{\pm}\mathcal{L}_{\pm}}{(\Omega''_{\pm})^2}. \tag{26}$$

Here the upper limit in the second integral b is determined by the next terms of expansion of frequency in δ, generally speaking $b=O(1)$. In Eq. (26) $\mathcal{L}_{\pm}\simeq\ln(\Omega''_{\pm}/\Gamma_{\pm})$. Finally, substituting Eqs. (24)–(26) into Eq. (16) we get the anisotropic GKE (7) with the collision term in the differential approximation

$$\mathrm{St}_{\boldsymbol k}(\{n(\boldsymbol{k}')\})=\mathrm{St}_{\boldsymbol{k},0}(\{n(\boldsymbol{k}')\})+\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\}). \tag{27}$$

Here $\mathrm{St}_{\boldsymbol{k},0}(\{n(\boldsymbol{k}')\})$ originates from $N_{\pm}^{(0)}$ and coincides with the collision term of KE [1]:

$$\mathrm{St}_{k,0}(\{n(\boldsymbol{k}')\})=\frac{A^2}{8\pi c}\left[\frac{1}{2}\int_0^k dq\, q^2(k-q)^2 N_-^{(0)}+\int_0^{\infty}dq\, q^2(k+q)^2 N_+^{(0)}\right]. \tag{28}$$

Term $\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\})$ is responsible for the angular evolution [and disappears for the isotropic distributions of $n(\boldsymbol{k})$, as expected]:

$$\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\})=\frac{A^2}{8\pi^2 c}\left[\frac{1}{2}\int_0^k dq\, q^2(k-q)^2\frac{\Gamma_-\mathcal{L}_-}{\Omega''_-}N''_-+\int_0^{\infty}dq\, q^2(k+q)^2\frac{\Gamma_+\mathcal{L}_+}{\Omega''_+}N''_+\right]. \tag{29}$$

III. WEAKLY ANISOTROPIC SPECTRA OF ACOUSTIC TURBULENCE

In this section we will find and analyze a steady state weakly anisotropic solution to the anisotropic GKE (7) with the collision term (28), (29) in the differential approximation

$$\mathrm{St}_{\boldsymbol{k},0}(\{n(\boldsymbol{k}')\})+\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\})=0. \tag{30}$$

A. Linearization of the basic equation

Due to the weak anisotropy of the problem, we are seeking a solution in the form

$$n(\boldsymbol{k})=n(k,\cos\theta)=n_0(k)+n_1(k,\cos\theta). \tag{31}$$

Here $n_0(k)$ is given by Eq. (4) and is a steady state solution of the isotropic problem

$$\mathrm{St}_{\boldsymbol{k},0}(\{n_0(k')\})=0. \tag{32}$$

The anisotropic correction is assumed to be small: $n_1(k,\cos\theta)\ll n_0(k)$. Then we substitute Eqs. (31)–(30) and linearize Eq. (30) by keeping only terms proportional to n_1 and discarding terms with higher orders of the correction. To this end, we expand the anisotropic correction $n_1(k,\cos\theta)$ in a series (10) in which $P_{\ell}(\cos\theta)$ is the Legendre polynomial of the order m, satisfying the equation

$$[\nabla_{\perp}^2+\ell(\ell+1)]P_{\ell}[\cos(\theta)]=0. \tag{33}$$

Consider first $\mathrm{St}_{\boldsymbol{k},0}(\{n(\boldsymbol{k}')\})$ given by Eq. (28). According to Eq. (32) $n_0(k)$ is the steady state solution of the GKE. Therefore the terms proportional to $n_0(k)$ have to disappear. The result (linear in n_1) is given by

$$\mathrm{St}_{k,0}(\{n(\boldsymbol{k}')\})=\sum_{\ell=1}^{\infty}P_{\ell}(\cos\theta)\Phi_0(k,f_{\ell}), \tag{34}$$

where

$$\begin{aligned}\Phi_0(k,f_{\ell}) &=\frac{A^2}{4\pi c}\Bigg[\frac{1}{2}\int_0^k dq\, q^2(k-q)^2\{f_{\ell}(k-q)[n_0(q)-n_0(k)]\\ &+n_0(k-q)[f_{\ell}(q)-f_{\ell}(k)]-f_{\ell}(k)n_0(q)-n_0(k)f_{\ell}(q)\}\\ &+\int_0^{\infty}dq\, q^2(k+q)^2\{f_{\ell}(k+q)\\ &\times[n_0(q)+n_0(k)]+n_0(k+q)[f_{\ell}(q)+f_{\ell}(k)]\\ &-f_{\ell}(k)n_0(q)-n_0(k)f_{\ell}(q)\}\Bigg].\end{aligned} \tag{35}$$

To linearize $\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\})$ we substitute in Eq. (29) the distribution $n(\boldsymbol{k})$ defined by Eqs. (4), (10), (31) and using Eq. (33) to get

$$\mathrm{St}_{\boldsymbol{k},2}(\{n(\boldsymbol{k}')\})=-\sum_{\ell=1}^{\infty}\ell(\ell+1)P_{\ell}[\cos(\theta)]\Phi_2(k,f_{\ell}), \tag{36}$$

where

$$\Phi_2(k,f_\ell)=\frac{A^2}{8\pi^2 c}\left(\frac{1}{2}\int_0^k dq\, q^2(k-q)^2\frac{\Gamma_-\mathcal{L}_-}{\Omega''_-}N''^{(\ell)}_- + \int_0^\infty dq\, q^2(k+q)^2\frac{\Gamma_+\mathcal{L}_+}{\Omega''_+}N''^{(\ell)}_+\right) \tag{37}$$

and

$$N''^{(\ell)}_\pm=\frac{1}{4}\{[n_0(q)\pm n_0(k)]f_\ell(k\pm q) + [n_0(k\pm q)-n_0(k)]f_\ell(q)\}.$$

The steady state weakly anisotropic GKE therefore reads

$$\sum_{\ell=1}^{\infty}[\Phi_0(k,f_\ell)-\ell(\ell+1)\Phi_2(k,f_\ell)]=0. \tag{38}$$

The particular solution of this equation which may satisfy any boundary conditions may be found only if each term in the sum vanishes, i.e.,

$$\Phi_0(k,f_\ell)=\ell(\ell+1)\Phi_2(k,f_\ell). \tag{39}$$

This is the basic equation for our study.

B. Evaluation of the collision integral

In order to solve Eq. (39) in the leading order on the class of scale invariant functions

$$f_\ell(q)=\frac{\phi_\ell}{q^x},\quad \phi_\ell \quad \text{is a prefactor}, \tag{40}$$

one has to evaluate the integrals in Φ_0 and Φ_2 and to find a leading contribution in the various regions of the exponents x. This is done in the Appendix, here we only will present and discuss the results.

As we show in Appendix 1, the leading contributions to Φ_0 (35) may be written as

$$\Phi_0\approx\frac{A^2 n_0(k)\phi_\ell(2x-9)}{2\pi c|(x-4)(x-5)|}\begin{cases}L^{x-5}, & x>5\\ k^{5-x}, & 4<x<5,\\ kk_*^{x-4}, & x<4.\end{cases} \tag{41}$$

For $x>3$ two integrals in Eq. (35) diverge in the IR regime (for $q\to 0$ and for $q\to k$). However, the leading divergent terms are canceled and the region of IR divergence becomes $x>4$. Moreover, first subleading terms also canceled and real region of IR divergence in Eq. (35) is $x>5$. In this regime one has to cut off the integral at outer scale L, see first line in Eq. (41). For $4<x<5$ the sum of integrals in Eq. (35) converges both in IR and ultraviolet (i.e., for $q\gg k$) regimes. The corresponding evaluation is given by the second line in Eq. (41). Finally, for $x<4$ the value of Φ_0 is dominated by the ultraviolet (UV) divergent contribution to the second integral in Eq. (35) and have to be regularized by some UV cutoff k_*. A corresponding evaluation is given by the last line in Eq. (41). The origin of k_* is different for different physical systems and will not be discussed here. Observe from Eq. (41) that (i) Φ_0 has definite sign, (ii) $\Phi_0=0$ at $x=9/2$, consistent with [1], (iii) prefactor in Φ_0 diverges for $x\to 5$ and $x\to 4$.

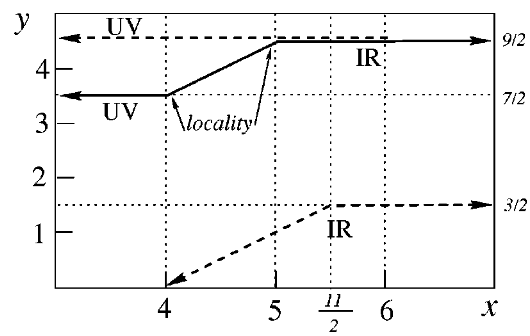

FIG. 2. Scaling exponents y_0 (solid line) and y_2 (dashed lines) for different values of x. Arrows show the directions of divergence for the corresponding integrals. IR and UV represent the infrared and ultraviolet limits, locality denotes the region of convergence of Φ_0.

In Appendix 2 we evaluated the leading contributions to integrals in Eq. (37) for Φ_2 for different values of x. The answers may be summarized as follows:

$$\Phi_2(k,f_\ell)\approx\frac{A^2\nu\mathcal{L}n_0(k)\phi_\ell}{2\pi^2c^2}\times\begin{cases}k^3L^{x-3} \quad (\text{for } 6<x),\\ k^3L^{x-3}+k_*^{6-x} \quad \text{for } \frac{11}{2}<x<6,\\ -(kL)^{5/2}k^{6-x}+k_*^{6-x} \quad \text{for } x<\frac{11}{2}.\end{cases} \tag{42}$$

In the IR regime we have two different contributions to Φ_2, the first is divergent for any x, the second is divergent for $x>3$. These contributions coincide at $x=11/2$. One may recognize the contributions from the IR regimes by the presence of L in the corresponding terms. In the integral (37) for Φ_2 we have UV divergence for $x<6$, in this region the upper cutoff again is k_* and corresponding terms involve this parameter. Note that in contrast to evaluations (41) we do not have here a window of locality where the collision integrals converge.

C. Solution of the homogeneous equation

Our goal now is to find the solution of the homogeneous Eq. (39). We seek the solution in the scale invariant sector. Denote the scaling exponents of $\Phi_0(k,f_\ell)$ and $\Phi_2(k,f_\ell)$ as y_0 and y_2:

$$\Phi_0(k,f_\ell)\propto\phi_\ell k^{-y_0},\quad \Phi_2(k,f_\ell)\propto\phi_\ell k^{-y_2}. \tag{43}$$

The x dependences of these exponents are determined by Eqs. (41), (42) and shown in Fig. 2.

Φ_0 converge for $4<x<5$ and diverge for other values of x (in IR for $5<x$ and in UV for $x<4$). Φ_2 never converge with different rate of divergence in IR and UV limits (two branches, see Fig. 2). The ratio between IR and UV divergent terms in Φ_2 depends on the wave vector k, therefore, generally speaking we have to account for both IR and UV terms.

Fortunately, the exponents $y_0 = y_2 = 9/2$ coincide in the window $5 < x < 6$, meaning that at least the k dependence of $\Phi_0(k, f_\ell)$ and $\Phi_2(k, f_\ell)$ is the same. Next promising observation is the coincidence of signs: both function are positive. It means that we have a chance to satisfy Eq. (39) by a proper choice of the exponent x. Indeed, according to Eqs. (41), (42), Eq. (39) for these values of x becomes

$$L^{x-5} = B'(x-5)\ell(\ell+1)\frac{\nu k_*^{6-x}}{c}, \qquad (44)$$

where B' is some positive, ℓ independent, dimensionless factor [we believe of $\mathcal{O}(1)$] which accumulated all unknown factors in our estimates.

Let us estimate effective viscosity ν. Substituting isotropic solution (4) in Eq. (14) one gets $N \simeq aL^{3/2}$. Now from Eq. (13) one has

$$\nu \simeq aL^{3/2}/\rho. \qquad (45)$$

Next one has to relate the dimensional factor a with the dimensionless amplitude of waves ϵ. To do this we evaluate the total energy of the acoustic waves

$$E_{ac} = \int d^3k\, \omega(k) n_0(k)$$
$$\simeq 4\pi c a \int_{1/L}^{\infty} \frac{dk}{k^{3/2}}$$
$$\simeq c a \sqrt{L}. \qquad (46)$$

We define ϵ^2 as the the ratio of the total energy in the acoustic-wave system E_{ac} to the total kinetic energy of the media E_{kin} which is about ρc^2. Parameter ϵ may be treated as the ratio of the amplitude of velocity in acoustic waves to the mean square velocity in the media caused by its kinetic energy in the thermodynamic equilibrium.

Now we have $\epsilon^2 \simeq a\sqrt{L}/\rho c$ which together with Eq. (45) gives

$$\nu \simeq \epsilon^2 c L. \qquad (47)$$

Using this evaluation in Eq. (44) one has

$$L^{x-6} = B''\epsilon^2 \ell(\ell+1) k_*^{6-x}, \qquad (48)$$

where B'' absorb one more unknown factor from the evaluation of ν. Therefore the solution is achieved for $x = x_{0,\ell}$, where

$$x_{0,\ell} = 6 + \frac{\ln[\epsilon^2 \ell(\ell+1)B'']}{\ln(k_* L)}. \qquad (49)$$

These exponents give the k dependence of the solution (39) in the depth of the inertial range. However, the functions ϕ_ℓ remain unknown. To find them one has to match the inertial range solution to the pumping at the IR boundary ($k = 1/L$).

D. Effect of inhomogeneous terms

Sometimes the inhomogeneous terms in the KE cause additional solutions which may play important role in the evolution of spectra in k. Consider first the origin of the inhomogeneous terms. Linearizing Eq. (34) we have concluded that n_0^2 contribution to $\text{St}_{\boldsymbol{k},0} = 0$ is zero. This is true only if the IR limit of the integral is indeed zero, which is not the case. In the energy containing interval $0 < k < L^{-1}$ we have a nonuniversal behavior of $n(\boldsymbol{k})$ which has to be accounted. A simple way to do this is first to evaluate the contribution of this region as

$$\text{St}_{\boldsymbol{k},0,\text{inhom}}(\{n(\boldsymbol{k}')\}) \simeq \frac{A^2 n_0(k)}{2\pi c\sqrt{L}} n\left(\frac{\boldsymbol{k}}{kL}\right). \qquad (50)$$

Note that in this region $n(\boldsymbol{k})$ is not isotropic and thus $\text{St}_{\boldsymbol{k},0,\text{inhom}}(\{n(\boldsymbol{k}')\})$ must depend on the direction of $\boldsymbol{k}$. Expanding this dependence into the spherical harmonics (in our case of the axial symmetry of the problem into the Legendre polynomials) we have an inhomogeneous contribution to Eq. (39):

$$\Phi_{\text{inhom}}(k, n_{0,\ell}) \approx \frac{A^2 n_0(k) n_{0,\ell}}{2\pi c\sqrt{L}} \propto k^z, \qquad (51)$$

where nonuniversal numbers $n_{0,\ell}$ may be related to ϕ_ℓ via ''boundary conditions'' at $k \approx 1/L$:

$$n_{0,\ell} \approx L^{x-9/2}\phi_\ell. \qquad (52)$$

An important observation is that the scaling exponent z of $\Phi_{\text{inhom}}(k, n_{0,\ell})$ in Eq. (51) is independent of x [because this term is independent of $f_\ell(k)$ altogether]. Moreover, $z = 9/2$, which is exactly the same as exponents y_0 and y_2 of the homogeneous part of the GKE Φ_0 and Φ_2. It means that for any k one has to account the contribution of $\Phi_{\text{inhom}}(k, n_{0,\ell})$ in the balance equation (39). Repeating the same calculations as in deriving Eq. (44) with the help of Eqs. (39), (41), and (42) and accounting now evaluations (51), (52) we are approaching again Eq. (44) with additional contribution to the factor B''; their sum may be denoted as B. Physically it means that resonant (i.e., with the same scaling exponent) inhomogeneous term has shifted the exponents x_ℓ relative to their ''homogeneous values'' $x_{0,\ell}$ (49). By replacing the unknown constants $B'' \Rightarrow B$ we obtained from Eq. (49) final equation (11) for the scaling exponents of x_ℓ.

IV. NON-SCALE-INVARIANT SOLUTIONS

In the previous section we found all scale-invariant solutions and observe that they have finite region of applicability for which their scaling exponents $5 < x_\ell < 6$. Outside of this region we have to look for non-scale-invariant solutions which contain explicitly a characteristic length scale Λ which will be chosen later. To do that we have to choose $f_\ell(k\Lambda)$ in some reasonable form, for example,

$$f_\ell(\kappa)=a_{0,\ell}\kappa^x\left[1+\frac{\alpha_{2,\ell}}{\kappa^2}+\frac{\alpha_{4,\ell}}{\kappa^4}\dots\right]\exp[-b_\ell\kappa^y],$$
$$\kappa\equiv k\Lambda\geqslant 1, \tag{53}$$

where x, y, $a_{i,\ell}$, and b_ℓ are some unknown numbers. Obviously there is no UV divergences in Φ_0 and Φ_2 for such choice of f_ℓ. Furthermore, since in *leading order* $\lim_{q\to 0}f_\ell=a_0$ there is no IR divergence in Φ_0 and Φ_2 associated with f_ℓ, so that the IR behavior of the Φ_0 and Φ_2 is dominated by $n_0(q_{\rm IR})$, $q_{\rm IR}\sim 1/L$.

Then in expression (35) we keep only the terms proportional to $n_0(q)$ and $n_0(k-q)$:

$$\Phi_0(k,f_\ell)\simeq\frac{A^2}{4\pi c}\left[\frac{1}{2}\int_{q_{\rm IR}}^k dq\, q^2(k-q)^2\{f_\ell(k-q)n_0(q)+n_0(k-q)[f_\ell(q)-f_\ell(k)]-f_\ell(k)n_0(q)\}+\int_{q_{\rm IR}}dq\, q^2(k+q)^2\{f_\ell(k+q)n_0(q)-f_\ell(k)n_0(q)\}\right]. \tag{54}$$

Now let us change variables in the first integral $q\to k-q$. Then we can evaluate integrals at $q=q_{\rm IR}$ only:

$$\Phi_0(k,f_\ell)\simeq\frac{A^2}{4\pi c}\left[\frac{1}{2}\int_{q_{\rm IR}}dq\, q^2(k-q)^2\{f_\ell(k-q)n_0(q)+n_0(k-q)[f_\ell(q)-f_\ell(k)]-f_\ell(k)n_0(q)+f_\ell(q)n_0(k-q)+n_0(q)\times[f_\ell(k-q)-f_\ell(k)]-f_\ell(k-q)n_0(k-q)\}+\int_{q_{\rm IR}}dq\, q^2(k+q)^2\{f_\ell(k+q)n_0(q)-f_\ell(k)n_0(q)\}\right]$$
$$\simeq\frac{A^2}{4\pi c}\left[\frac{1}{2}\int_{q_{\rm IR}}dq\, q^2k^2 2n_0(q)[f_\ell(k+q)-2f_\ell(k)+f_\ell(k-q)]\right]$$
$$\simeq\frac{A^2}{8\pi c}\frac{k^2\partial^2 f_\ell(k)}{\partial k^2}\int_{q_{\rm IR}}dq\, q^4 n_0(q). \tag{55}$$

The second derivative of f_ℓ is the result of the usual double cancellation of the IR divergence in the isotropical part of KE.

Now we have to evaluate Φ_2 at $q\ll k$ for the choice of f_ℓ given by Eq. (53). Let us first evaluate N''_- for $q\ll k$:

$$N''^{(\ell)}_-\simeq\frac{1}{4}\{n_0(q)f_\ell(k-q)+[n_0(k-q)-n_0(k)]f_\ell(q)\} \tag{56}$$

or

$$N''^{(\ell)}_-\simeq\frac{1}{4}n_0(q)f_\ell(k),\quad q\ll k. \tag{57}$$

Similarly,

$$N''^{(\ell)}_+\simeq\frac{1}{4}n_0(q)f_\ell(k),\quad q\ll k. \tag{58}$$

Now use Eq. (A6) to estimate Φ_2:

$$\Phi_2(k,f_\ell)\simeq\frac{A^2}{16\pi^2 c}\int_{q_{\rm IR}}dq\, q^2k^2\frac{4\nu k^2\tilde{\mathcal{L}}}{cq}n_0(q)f_\ell(k)$$
$$\simeq\frac{A^2}{16\pi^2 c}q_{\rm IR}^2k^4 4\epsilon^2 L n_0(q_{\rm IR})f_\ell(k),\quad q_{\rm IR}\simeq 1/L, \tag{59}$$

where we used estimation (47) for effective viscosity ν. Actually we have not calculated accurately the $q\to k$ contribution in Φ_2. Note, however, that this contribution would be of lower (in q) order, because $\lim_{q\to k}\simeq-[ck^2/2(k-q)]$.

Now Eq. (39) acquires the form (introducing dimensionless constant C, generally speaking of the order of unity)

$$q_{\rm IR}^5k^2n_0(q_{\rm IR})\frac{d^2f_\ell(k)}{dk^2}=C\epsilon^2L\ell(\ell+1)q_{\rm IR}^2k^4n_0(q_{\rm IR})f_\ell(k). \tag{60}$$

Note that $n_0(q_{\rm IR})$ cancels from both sides of this equation. Further notice that both terms are of positive sign. Canceling $q_{\rm IR}$ and k from both sides and substituting $\kappa=k/\Lambda$ with $q_{\rm IR}\simeq 1/L$ we get

$$\frac{d^2f_\ell(\kappa)}{d\kappa^2}\simeq C\ell(\ell+1)\epsilon^2\kappa^2f_\ell(\kappa),\quad \kappa\equiv kL, \tag{61}$$

where we chose $\Lambda\simeq L$. This equation may be solved in the Bessel functions (of order 1/4 of imaginary argument). For our goals, however, it would be enough to present only its asymptotic form for $\kappa\gg 1$:

$$f_\ell(\kappa)=\sqrt{\kappa}\left[1+\frac{a_{2,m}}{\kappa^2}+\cdots\right]\exp\left(-\frac{\epsilon}{2}\sqrt{C\ell(\ell+1)}\kappa^2\right), \tag{62}$$

where coefficients $a_{2,\ell}$, $a_{4,\ell}$, etc., may be found iteratively: $a_{2,\ell}=3/\sqrt{C\ell(\ell+1)\epsilon^2},\dots$, but these terms really are beyond the accuracy of Eq. (61) itself. Solution (62) shows that anisotropic corrections decay exponentially, the rate of decay increases with ϵ^2 and $\ell(\ell+1)$.

V. SUMMARY

Let us summarize the logic of this paper. We start from generalized kinetic equation (GKE) (8). The conditions of time-space resonances (1) dictates almost collinear propagation, so we expand Eq. (8) in the transverse direction, and arrive to the differential approximation for the collision term in GKE (27) with Eqs. (28) and (29). We assume *weak* anisotropy with anisotropic correction in the factorized form (31). After substitution (31) in Eqs. (27), (28), (29), we *linearize* resulting three-dimensional equations (which are differential in angles and integral along the rays). Expansion of these equations in series in Legendre polynomials P_ℓ foliates the three-dimensional problem in decoupled sets on one-dimensional ones, for each ℓ separately. We have found the power-law solutions of these equations (11) and the exponential solutions (12) which are governed by the same parameter $\epsilon^2\ell(\ell+1)$. These solutions describe the phenomenon of isotropization of nondispersive acoustic turbulence in the cascade process of energy transfer from anisotropic region of pumping down to the smaller and smaller scales $(k\to\infty)$. In the limit $k\to\infty$ the statistics of acoustic turbulence approaches isotropical flux equilibrium. The main result of the article is that the spectra of acoustic turbulence tend to become more isotropic.

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APPENDIX: ANALYSIS OF CONVERGES OF INTEGRALS IN THE COLLISION TERMS

1. $\Phi_0(k,f_\ell)$ behavior

a. IR regime

In Eq. (35) one has three different regions dangerous for IR divergence. These are the regions $q\to 0$ and $q\to k$ in the first integral and $q\to 0$ in the second one. The first integral $1/2\int_0^k\cdots$ may be split into two integrals $1/2\int_0^{k/2}\cdots$ and $1/2\int_{k/2}^k\cdots$. In the second one we may change dummy variable $q\to q'=(k-q)$ and then to redenote $q'\to q$. After that one sees that the integral $1/2\int_{k/2}^k\cdots$ equal to $1/2\int_0^{k/2}\cdots$ and therefore the first integral in Eq. (35), $1/2\int_0^k\cdots$, may be rewritten as $\int_0^{k/2}\ldots$. This integral in the region $q\ll k$ together with the second integral in Eq. (35) may be written as

$$\Phi_0(k,f_\ell)\approx\frac{A^2}{4\pi c}\int_0^{k/2}dq\,q^2(k-q)^2 f_\ell(q)\times[n_0(k-q)-n_0(q)]+(k+q)^2 f_\ell(q)[n_0(k+q)-n(k)]. \quad \text{(A1)}$$

Substituting $f_\ell(q)$ from Eq. (40) one observes here a usual double cancellation of the leading and the first subleading terms in the IR regime. Therefore integral (A1) may be evaluated (up to a factor) as follows:

$$\Phi_0(k,f_\ell)\approx\frac{A^2 n_0(k)}{4\pi c}\int_0^{k/2}dq\,q^4 f_\ell(q). \quad \text{(A2)}$$

Now one sees that the integral (A2) converges for $x<5$ and diverges for $x>5$. In the latter case we take the outer scale L as an IR cutoff. Then it is easy to see that Φ_0 may be evaluated in different regions of x as

$$\Phi_0\approx\frac{A^2 n_0(k)\phi_\ell}{2\pi c}\times\begin{cases} k^{5-x}/(5-x) & (\text{for } x<5),\\ \ln(kL) & (\text{at } x=5),\\ L^{x-5}/(x-r) & (\text{for } 5<x).\end{cases} \quad \text{(A3)}$$

Note, that in this limit and for x close to 5, $\Phi_0>0$.

b. UV regime

Next we analyze the region $q\gg k$, i.e., the UV regime in the second integral in Eq. (35). Here the most dangerous terms reduce to

$$\Phi_0(k,f_\ell)\approx\frac{A^2 n_0(k)}{4\pi c}\int_k^{k_*}dq\,q^4[f_\ell(k+q)-f_\ell(q)]. \quad \text{(A4)}$$

There is a cancellation of the leading contribution in the region $q\gg k$. Therefore this integral converges at $x>4$. In a compact form the UV evaluation of Φ_0 may be written as

$$\Phi_0\approx-\frac{A^2 k n_0(k)\phi_\ell}{2\pi c|x-4|}\times\begin{cases} k^{4-x}, & x>4\\ k_*^{4-x}, & x<4.\end{cases} \quad \text{(A5)}$$

Here k_* is an UV cutoff of the integral which may have different nature for different physical situations and will not be further clarified here. Note, that in the UV regime for x close to 4, $\Phi_0<0$. It is known also, that $\Phi_0=0$ for $x=9/2$, the scaling index of the isotropic solution of the KE. Now we can join Eqs. (A3) and (A5) and write evaluation of the leading contribution to $\Phi_0=0$ in all regions of x in the form (41).

2. $\Phi_2(k,f_\ell)$ behavior

a. IR regime

Unlike $\Phi_0(k,f_\ell)$, there is no double cancellation of the IR contribution (i.e., in the limits $q\to 0$ and $q\to k$) in the sum of the integrals (37) for $\Phi_2(k,f_\ell)$. The reason is that $\Omega''_\pm$ is not invariant under $q\to k-q$ transformation. Now, in the IR region $q\ll k$ one has the following simplifications:

$$\Gamma_\pm=2\nu(k^2+k_1^2\pm kk_1)\approx 2\nu k^2,$$

$$\Omega_\pm\simeq\delta^2\Omega''_\pm,\quad \Omega''_\pm\equiv\pm\frac{ckk_1}{2(k\pm k_1)}\simeq\pm cq/2,$$

$$\mathcal{L}_{\pm}=\ln\left(\frac{\Omega''_{\pm}}{\Gamma_{\pm}}\right)\simeq\ln\left(\frac{cq}{4\nu k^2}\right)\equiv\tilde{\mathcal{L}}. \tag{A6}$$

Substituting these equations into Eq. (37) one gets in the IR region

$$\Phi_2(k,f_{\mathscr{l}})\simeq\frac{\nu k^4A^2\tilde{\mathcal{L}}}{2\pi^2c^2}\int_0^k dq\,q\left[\frac{1}{2}N''^{(\mathscr{l})}_{-}-N''^{(\mathscr{l})}_{+}\right]. \tag{A7}$$

The most dangerous terms in the combinations $N''^{(\mathscr{l})}_{\pm}$ are

$$N''^{(\mathscr{l})}_{\pm}\simeq\frac{1}{4}\{n_0(q)f_{\mathscr{l}}(k\pm q)+[n_0(k\pm q)-n_0(k)]f_{\mathscr{l}}(q)\}. \tag{A8}$$

Together with Eq. (32) these yield

$$N''^{(l)}_{\pm}\simeq\frac{1}{4}\left\{n_0(q)f_l(k)\mp\frac{9q}{2k}n_0(k)f_{\mathscr{l}}(q)\right\}. \tag{A9}$$

Then in the IR limit Φ_2 reads

$$\Phi_2(k,f_{\mathscr{l}})\simeq\frac{\nu k^4A^2\tilde{\mathcal{L}}}{\phi^2c^2}\int_0^k dq\,q \times\left[-n_0(q)f_{\mathscr{l}}(k)+\frac{q}{k}n_0(k)f_{\mathscr{l}}(q)\right], \tag{A10}$$

where we did not care about the numerical factor, just carrying the signs. Since $n_0(k)\propto k^{-9/2}$ the $n_0(q)f_{\mathscr{l}}(k)$ term always diverges; the corresponding contribution to the integral behaves as $(kL)^{5/2}k^2n_0(k)f_{\mathscr{l}}(k)$. The second term will diverge if $f_{\mathscr{l}}(q)=\phi_{\mathscr{l}}/q^x$ with $x<3$. Symbolically, its contributions to the integral are

$$\int_0\cdots\sim\frac{n_0(k)\phi_{\mathscr{l}}}{k|x-3|}\begin{cases}k^{3-x} & (\text{for } x<3),\\ L^{x-3} & (\text{for } x>3).\end{cases} \tag{A11}$$

Observe that for $x>11/2$ the contribution of the second term in Eq. (A10) dominates the contribution of the first one. Finally, the IR contributions to $\Phi_2(k,f_{\mathscr{l}})$ are summarized as

$$\Phi_2(k,f_{\mathscr{l}})\approx\frac{\nu k^3A^2\mathcal{L}n_0(k)\phi_{\mathscr{l}}}{2\pi^2c^2}\times\begin{cases}-(kL)^{5/2}k^{3-x} & (\text{for } x<11/2),\\ L^{x-3} & (\text{for } x>11/2).\end{cases} \tag{A12}$$

b. UV-regime

Next we have to consider the UV regime of the integral in Eq. (37), i.e., the region of integration with $q\to k_{\mathscr{l}}\gg k$. Here we have the simplifications

$$\Gamma_{+}=2\nu(k^2+k_1^2\pm kk_1)\simeq2\nu q^2, \tag{A13}$$

$$\Omega_{+}\simeq\delta^2\Omega''_{\pm},\quad \Omega^{IJ}_{\pm}\equiv\pm\frac{ckk_1}{2(k\pm k_1)}\cong\frac{ck}{2},$$

$$\mathcal{L}_{+}=\ln\left(\frac{\Omega''_{\pm}}{\Gamma_{\pm}}\right)\simeq\ln\left(\frac{ck}{4\nu q^2}\right)\equiv\hat{\mathcal{L}}.$$

Now Φ_2 becomes

$$\Phi_2(k,f_{\mathscr{l}})=\frac{A^2}{8\pi^2c}\int_k^{k_*}dq\,q^2(k+q)^2\frac{\Gamma_{+}\mathcal{L}_{+}}{\Omega''_{+}}N''^{(\mathscr{l})}_{+} \simeq\frac{A^2\nu\tilde{\mathcal{L}}}{2\pi^2c_2k}\int_k^{k_*}dq\,q^6N''^{(\mathscr{l})}_{+}. \tag{A14}$$

Out of $N''^{(\mathscr{l})}_{+}$ the most divergent for UV term is

$$N''^{(\mathscr{l})}_{+}\simeq\frac{1}{4}\{n_0(k)[f_{\mathscr{l}}(k+q)-f_{\mathscr{l}}(q)]\} \sim-\frac{xk}{4q}n_0(k)f_{\mathscr{l}}(q), \tag{A15}$$

where we used that $f_{\mathscr{l}}(q)\propto q^{-x}$ and $q\gg k$. Now in the UV regime the $\Phi_2(k,f_{\mathscr{l}})$ term may be evaluated (up to a numerical factor) as

$$\Phi_2(k,f_{\mathscr{l}})\simeq-\frac{A^2\nu\tilde{\mathcal{L}}n_0(k)}{2\pi^2c^2}\int_k^{k_*}dq\,q^5f_{\mathscr{l}}(q). \tag{A16}$$

This integral converges for $x>6$ and may be written as follows:

$$\Phi_2(k,f_{\mathscr{l}})\sim-\frac{A^2\nu\tilde{\mathcal{L}}n_0(k)\phi_{\mathscr{l}}}{2\pi^2c^2|x-6|}\times\begin{cases}k_*^{6-x} & \text{for } x<6,\\ k^{6-x} & \text{for } x>6.\end{cases} \tag{A17}$$

Combining Eqs. (A17) and (A12), one has after some minor manipulations (neglecting factors of order 1, difference between $\mathcal{L}$ and $\hat{\mathcal{L}}$, etc.) Eq. (42).

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